

HOW UNSTABLE ARE FUNDAMENTAL QUANTUM SUPERMEMBRANES?

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ABSTRACT

1 Quantum Supermembranes

String duality, pioneered by Kikkawa and Yamasaki^{1,2}, represents an enormous advance in our understanding of string physics. For the first time, we can peer into the non-perturbative region of certain string theories and settle questions which have dogged the field since its very inception³⁻⁸.

In particular, solitonic p -branes are necessary to complete our understanding of BPS-saturated states. An eleven dimensional “M-theory,” in fact, must be able to incorporate both strings and solitonic membranes. Because these solitonic membranes likely have finite thickness, they are probably stable.

By contrast, fundamental quantum supermembranes are thought to have serious problems. Besides the fact that they are highly non-linear (and hence their spectrum is impossible to calculate exactly), they also have several serious physical diseases:

- (a) the world volume action is not renormalizable
- (b) the theory has no dilaton, so a standard KSV type perturbation theory is not possible

(c) the theory is thought to be unstable; for string-like configurations, the zero-point energy of the Hamiltonian may be zero⁹.

The first problem means that an infinite number of counter-terms must be added to the world volume action to render it finite. However, perhaps these counter-terms simply represent the infinite number of background fields corresponding to excitations of the supermembrane. So having an infinite number of counter-terms is by itself not necessarily fatal.

The second problem is also not necessarily fatal if a new mechanism is found for interacting membranes, other than the standard dilaton formulation. Since we do not know how supermembranes split apart (or even if they do), it is premature to discount them on the basis of interactions.

The third problem is more serious, since it goes to the heart of whether fundamental quantum supermembranes are stable or not.

Previously, in ref. 9, this question was addressed by approximating the supermembrane action^{10,11} by a $SU(n)$ super Yang-Mills theory as $n \rightarrow \infty$. For finite n , this amounts to a convenient regulator for the theory. Although this proof is rather convincing, it depends on whether the $n \rightarrow \infty$ limit is singular or not. Perhaps there are regularization-dependent factors which enter into the picture in this delicate limit.

In this paper, we will try to address the question directly, whether the continuum theory is stable or not. By analyzing the continuum theory, we have a much more intuitive grasp of precisely where the problems may lie and where the potential infinities may occur. We will follow the basic outline of ref. 9, but adapt their calculation for our purposes.

And second, at the end of this paper, we present some rough speculations about how unstable membranes may still be made into a physical theory.

We begin with the action for the membrane, which is given by:

$$S = S_1 + S_2 \tag{1}$$

where S_1 is the usual determinant defined over a world volume:

$$S_1 = -T \int d^3\sigma \sqrt{-M}; \quad M_{ij} = \Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu} \tag{2}$$

where:

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\theta}\Gamma^\mu\partial_i\theta \tag{3}$$

and S_2 is a Wess-Zumino term^{10,11}:

$$S_2 = -T \int d^3\sigma \left[\frac{1}{2} \epsilon^{ijk} \bar{\theta} \Gamma_{\mu\nu} \partial_i \theta \left(\Pi_j^\mu \Pi_k^\nu + i \Pi_j^\mu \bar{\theta} \Gamma^\nu \partial_k \theta - \frac{1}{3} \bar{\theta} \Gamma^\mu \partial_j \theta \bar{\theta} \Gamma^\nu x \partial_k \theta \right) \right] \quad (4)$$

where $i = 1, 2, 3$ represents the three world volume indices of the membrane. Two of them, σ_1 and σ_2 , represent the co-ordinates of the surface, and $\sigma_3 = \tau$ represents the time-like direction. The Greek symbols represent 11 dimensional Lorentz indices. Γ^μ are the usual Dirac matrices in 11 dimensions. X^μ is the co-ordinate of the membrane, and θ is a Majorana spinor with 32 real components.

This action is invariant under a standard reparametrization invariance:

$$\delta X^\mu(\sigma_1, \sigma_2, \sigma_3) = \epsilon^i \partial_i X^\mu(\sigma_1, \sigma_2, \sigma_3) \quad (5)$$

The Majorana spinor θ also transforms as a scalar under reparametrizations in the world volume variables.⁴

Under local supersymmetry, we have:

$$\delta X^\mu = \bar{\theta} \Gamma^\mu (1 + \Gamma) \kappa; \quad \delta \theta = (1 + \Gamma) \kappa \quad (6)$$

where κ is a local parameter, and:

$$\Gamma = \frac{1}{6\sqrt{-g}} \epsilon^{ijk} \Pi_i^\mu \Pi_j^\nu \Pi_k^\rho \Gamma_{\mu\nu\rho} \quad (7)$$

and where $\Pi_i \cdot \Pi_j = g_{ij}$.

The action as it stands is intractable because of its highly coupled nature. The simplest way of simplifying and quantizing the theory is to go to the light cone gauge, where all longitudinal modes are removed. We impose:

$$\Gamma^+ \theta = 0 \quad (8)$$

along with the usual bosonic constraints. A large number of terms vanishes in the light cone gauge because $\bar{\theta} \Gamma^\mu \partial_i \theta = 0$ except for $\mu = -$. In particular, the higher order coupled terms of the action disappear in this gauge.

Then the reduced equations of motion can be derived from the Hamiltonian:

$$H = \int d^2\sigma \left[\frac{1}{2}(P^I)^2 + \frac{1}{4}(\{X^I, X^J\})^2 - \frac{i}{2}\bar{\theta}\Gamma^I\{X^I, \theta\} \right] \quad (9)$$

where $I = 1, 2, \dots, 9$ and:

$$\{A, B\} = \partial_1 A \partial_2 B - (1 \leftrightarrow 2) \quad (10)$$

and where the physical states are constrained by:

$$\{\dot{X}^I, X^I\} + \{\bar{\theta}, \theta\} = 0 \quad (11)$$

which vanishes on physical states. This constraint generates area preserving diffeomorphisms.

The problem with this Hamiltonian is that, for certain configurations of the membrane, the potential function, which is the second term in the Hamiltonian (9), vanishes. This is potentially disastrous for the theory. Let $f(\sigma_1, \sigma_2)$ represent a function of the membrane variables, and consider $X_\mu(f)$, which represents a string-like configuration. For this string-like configuration, the potential function disappears because:

$$\{X_\mu(f), X_\nu(f)\} = 0 \quad (12)$$

This means that classically, the potential function of the bosonic Hamiltonian vanishes along string-like filaments with zero area that protrude from the membrane like the quills of a porcupine. In principle, this may destabilize the Hamiltonian, allowing leakage of the wave function along these strings. In ref. 9, the potential was shown to vanish when X was approximated by fields defined in the Cartan sub-algebra of $SU(n)$. Because the elements of the Cartan sub-algebra commute among each other, the potential term was shown to vanish.

However, it is not obvious that this means that the theory is unstable along these string-like configurations. Let us study a toy-model to understand the subtleties of the question.

As in ref. 12, let us begin with a simple quantum mechanical system in two dimensions, with the potential given by x^2y^2 :

$$H_B = -\Delta + x^2y^2 \quad (13)$$

This Hamiltonian resembles the supermembrane theory because the interaction Hamiltonian vanishes along the x and y axes, so naively one may expect that the wave function can “leak” along the axes and the theory is therefore unstable. However, this is not true. Let us temporarily fix the value of x , which is defined to be large. If we move a short distance along the y axis, the potential function is a potential well for the harmonic oscillator which is quite steep for large values of x , so the leakage is quite small. For large x , the leakage is infinitesimally small. So which effect dominates?

In fact, the spectrum is actually discrete. For fixed x , the Hamiltonian obeys $H_B \geq |x|$, so the energy necessary to move the wave function to infinity is infinite. In fact,

$$H_B \geq (|x| + |y|)/2 \tag{14}$$

so the spectrum is discrete.

This toy model shows that there are subtleties with regard to the stability of even simple quantum mechanical systems. However, the theory can still become unstable if we introduce fermions and supersymmetry. The zero point energy from the fermions can cancel the $|x|$ contribution, giving us an unstable theory.

Start with the quantum mechanical system⁹:

$$H = \frac{1}{2}\{Q, Q^\dagger\} \tag{15}$$

where:

$$Q = Q^\dagger = \begin{pmatrix} -xy & i\partial_x + \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix} \tag{16}$$

The Hamiltonian reads:

$$H = \begin{pmatrix} -\Delta + x^2y^2 & x + iy \\ x - iy & -\Delta + x^2y^2 \end{pmatrix} \tag{17}$$

For fixed $|x|$, the supersymmetry of the reduced system is enough to guarantee that the energy contribution coming from the fermionic variables cancels the contribution from the bosonic variables. In fact, if we define:

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (18)$$

then:

$$\xi^T H \xi = H_B - x \quad (19)$$

so the fermionic contribution cancels the x coming from the bosonic variables, and the system becomes unstable.

We can introduce normalized wave functions for this case as:

$$\Psi_t(x, y) = \chi(x - t) \pi^{-1/4} |x|^{1/4} \exp\left(-\frac{1}{2}|x|y^2\right) \xi \quad (20)$$

for the ground state. t is a parameter which will be taken to be arbitrarily large; it measures the distance we have shifted the wave function along the x direction. χ is a function which has compact support. Then we can see that:

$$\lim_{t \rightarrow \infty} (\Psi_t, H^n \Psi_t) = \int dx \chi(x)^* (-\partial_x^2)^n \chi(x) \quad (21)$$

for $n = 0, 1, 2$, so we can shift the wave function as t goes to infinite without having to supply an infinite amount of energy. In fact, if E is the energy of this system, we can see that E can have any arbitrary value, corresponding to the eigenvalue of $-\partial_x^2$, where the potential vanishes. Hence, the energy spectrum is continuous.

A similar situation may happen with quantum supermembranes. Naively, the bosonic membrane theory seems to be unstable because the potential vanishes along certain string-like directions. However, the amount of energy necessary for the wave function to leak along these directions is infinite. But when we add fermions into the theory, then we must check explicitly if the fermionic contribution to the zero point energy cancels the bosonic contribution.

In ref. 9, this was studied by approximating the membrane with super Yang-Mills theory. We wish, however, to keep the continuum limit throughout, and at the very last step identify where any infinities may arise and where regularization methods may be necessary.

2 Zero Point Energy

Now let us calculate the zero point energy for the quantum supermembrane in the light cone gauge. Let us divide the original X^I membrane co-ordinate into several parts. Let x represent the co-ordinate along the string, so that:

$$x = x(f(\sigma_1, \sigma_2)) \quad (22)$$

We will let Y be the co-ordinate of the membrane which lies off the string, i.e. it cannot be written as a function of a single string variable.

In order to carry out gauge fixing, let us select out the 9th co-ordinate from I . Let the a index represent $1, 2, \dots, 8$.

Now let us split the original X^I into different pieces. Not only will we split the 9th component off from the others, we will also explicitly split X^I into $x(f)$ and Y .

Then:

$$X^I = (x_9, Y_9, x_a, Y_a) \quad (23)$$

(At the end of the calculation, we will shift along the string-like configuration as some parameter $t \rightarrow \infty$, where $x_9 = x(f)$ grows like t , while x_a goes to a constant. So x_a can be dropped in relation to x , but we will keep both variables in our equations until the very last step.)

We can fix the gauge by choosing $Y_9 = 0$.

Then the Hamiltonian can be split up into several pieces:

$$H = H_1 + H_2 + H_3 + H_4 \quad (24)$$

where:

$$\begin{aligned} H_1 &= -\frac{1}{2} \int d^2\sigma \left[\left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial x_a} \right)^2 \right] \\ H_2 &= -\frac{1}{2} \int d^2\sigma \left(\frac{\partial}{\partial Y_a} \right)^2 + \frac{1}{2} \int d^2\sigma d^2\bar{\sigma} d^2\sigma' \left[Y_a(\bar{\sigma}) z^T(\bar{\sigma}, \sigma') z(\sigma', \sigma) Y_a(\sigma) \right] \\ H_3 &= -\frac{i}{2} \int d^2\sigma d^2\bar{\sigma} \bar{\theta}(\bar{\sigma}) \left[z(\bar{\sigma}, \sigma) \gamma_9 + z_a(\bar{\sigma}, \sigma) \gamma_a \right] \theta(\sigma) \end{aligned} \quad (25)$$

where:

$$\begin{aligned} z(\bar{\sigma}, \sigma) &= \delta^2(\bar{\sigma}, \sigma) \partial_{\sigma_1} x \partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \\ z_a(\bar{\sigma}, \sigma) &= \delta^2(\bar{\sigma}, \sigma) \partial_{\sigma_1} x_a \partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \end{aligned} \quad (26)$$

and the index σ is shorthand for $\{\sigma_1, \sigma_2\}$. Notice that $z(\bar{\sigma}, \sigma)$ is an anti-symmetric function. Also, we have set $\Gamma_a = \gamma)a$. H_4 contains other terms in Y , which will not concern us yet.

The key factor, which will dominate our entire discussion of the zero point energy, is $z(\sigma, \bar{\sigma})$, which is the continuous matrix element which defines the diffeomorphism algebra in equation (10). In particular, we are interested in the sub-algebra of $w(\infty)$ which defines the reparametrization along the string-like filament. For elements $x(f)$, the elements of the algebra commute among each other. (In ref. 9, the counterpart of $x(f)$ are elements of the Cartan sub-algebra, which commute among each other by definition.) z is important to our discussion because the zero point energy can be defined entirely in terms of its eigenvalues.

Now consider the term H_2 . We can write down an eigenfunction for H_2 as:

$$\Phi_0 = A(\det \Omega)^2 \exp\left(-\frac{1}{2} \int d^2 \bar{\sigma} d^2 \sigma Y_a(\bar{\sigma}) \Omega(\bar{\sigma}, \sigma) Y_a(\sigma)\right) \quad (27)$$

where Ω is yet undetermined, and A is a normalization constant, determined by:

$$1 = (\Phi_0, \Phi_0) = \int \prod_a \prod_{\sigma} \mathcal{D}Y_a(\sigma) \Phi_0^* \Phi_0 \quad (28)$$

Applying H_2 to this wave function, we find:

$$H_2 \Phi_0 = 4 \int d^2 \sigma \Omega(\sigma, \sigma) \Phi_0 \quad (29)$$

which fixes the value of Ω to be:

$$\Omega^2(\bar{\sigma}, \sigma) = \int d^2 \sigma' z^T(\bar{\sigma}, \sigma') z(\sigma', \sigma) \quad (30)$$

To find an explicit expression for the ground state energy requires that we take the trace of Ω . This is a tricky problem, since the trace may actually

diverge, requiring a regularization. Let us assume that we can diagonalize the z by finding its eigenvalues. Let us introduce eigenvectors E_{MN}^σ , where $M \neq N$, as follows:

$$\int d^2\sigma z(\bar{\sigma}, \sigma) E_{MN}^\sigma = i\lambda_{MN} E_{MN}^\sigma \quad (31)$$

$$\int d^2\sigma z_a(\bar{\sigma}, \sigma) E_{MN}^\sigma = i\lambda_{MN}^a E_{MN}^\sigma \quad (32)$$

where M, N label a complete set of orthonormal functions, which can be either continuous or discrete, and λ_{MN} are the anti-symmetric eigenvalues of z . Our discussion will not depend on the explicit representation. (Since z and z_a commute, we can diagonalize them with the same eigenvectors.)

We can normalize them as follows:

$$\int d^2\sigma (E_{MN}^\sigma)^* E_{PQ}^\sigma = \delta_{MP}\delta_{NQ} \quad (33)$$

$$\sum_{M \neq N} (E_{MN}^\sigma)^* E_{MN}^{\bar{\sigma}} = \delta^2(\bar{\sigma} - \sigma) \quad (34)$$

$$(E_{MN}^\sigma)^* = E_{NM}^\sigma \quad (35)$$

If we diagonalize z in terms of these eigenvalues, we find that the eigenvalue of H_2 is given by the sum of the absolute values of the eigenvalues of z :

$$\int d^2\sigma \Omega(\sigma, \sigma) = \sum_{M, N} |\lambda_{MN}| \quad (36)$$

$$\det \Omega = \prod_{M < N} \lambda_{MN}^2 \quad (37)$$

(Because λ_{MN} is anti-symmetric, we can reduce the product over all M, N to one with $M < N$, where the precise ordering of the indices is arbitrary.) Now let us calculate the contribution of the fermionic variables to the zero point energy.

3 Fermionic Variables

The calculation of the fermionic variables is more difficult. As before our plan is to express all quantities in terms of the eigenvalues of the matrix $z(\bar{\sigma}, \sigma)$. Our calculation will resemble the path taken in ref. 9.

We now change variables to:

$$\theta(\sigma) = \sum_{M \neq N} \theta^{MN} E_{MN}^\sigma \quad (38)$$

The original fermionic variables are real. This means, therefore, that:

$$\theta^{MN\dagger} = \theta^{NM} \quad (39)$$

We can check that the anti-commutation relations:

$$[\theta_\alpha(\sigma), \theta_\beta(\bar{\sigma})]_+ = \delta_{\alpha,\beta} \delta^2(\sigma - \bar{\sigma}) \quad (40)$$

are transformed into:

$$[\theta_\alpha^{MN}, \theta_\beta^{PQ}]_+ = \delta_{\alpha,\beta} \delta^{MQ} \delta^{NP} \quad (41)$$

The fact that we can convert the complex θ^{MN} into its conjugate by simply reversing the lower indices is quite convenient, but it will allow us to establish creation and annihilation operators.

Then H_3 reduces to:

$$\begin{aligned} H_3 &= \frac{1}{2} \sum_{M \neq N} \theta^{NM} (\lambda_{MN} \gamma_9 + \lambda_{MN}^a \gamma_a) \theta^{MN} \\ &= \sum_{M < N} \theta^{MN\dagger} (\lambda_{MN} \gamma_9 + \lambda_{MN}^a \gamma_a) \theta^{MN} \end{aligned} \quad (42)$$

Now let us make a change in fermionic variables to eliminate the presence of γ_a in the above expression. Let us define:

$$\tilde{\theta}^{MN} = (A_{MN} + B_{MN}^a \gamma_a \gamma_9) \theta^{MN} \quad (43)$$

When we insert this expression back into the one for H_3 , we demand that H_3 reduce down to a function of $\tilde{\theta}^{MN\dagger}\gamma_9\tilde{\theta}^{MN}$. We then find a system of two equations:

$$\begin{aligned}\omega_{MN} [A_{MN}^2 - (B_{MN}^a)^2] &= \lambda_{MN} \\ -2A_{MN}B_{MN}^a\omega_{MN} &= \lambda_{MN}^a\end{aligned}\quad (44)$$

whose solutions are given by:

$$\begin{aligned}A_{MN} &= \frac{1}{\sqrt{2\omega_{MN}}}\sqrt{\omega_{MN} + \lambda_{MN}} \\ B_{MN}^a &= -\frac{1}{\sqrt{2\omega_{MN}}}\frac{\lambda_{MN}^a}{\sqrt{\omega_{MN} + \lambda_{MN}}} \\ \omega_{MN} &= \sqrt{\lambda_{MN}^2 + (\lambda_{MN}^a)^2}\end{aligned}\quad (45)$$

So the expression for H_3 reduces to:

$$H_3 = \sum_{M,N} \omega_{MN} \tilde{\theta}^{MN\dagger} \gamma_9 \tilde{\theta}^{MN} \quad (46)$$

Lastly, in order to eliminate the presence of γ_9 , let us introduce projection operators:

$$P_{\pm} = \frac{1 \pm \gamma_9}{2} \quad (47)$$

so that:

$$\phi\gamma_9\theta = \phi_+\theta_+ - \phi_-\theta_- \quad (48)$$

Then H_3 becomes:

$$\begin{aligned}H_3 &= \sum_{M<N} (\tilde{\theta}_+^{MN\dagger}\tilde{\theta}_+^{MN} - \tilde{\theta}_-^{MN\dagger}\tilde{\theta}_-^{MN}) \\ &= \sum_{M<N} (\tilde{\theta}_+^{MN\dagger}\tilde{\theta}_+^{MN} + \tilde{\theta}_-^{MN}\tilde{\theta}_-^{MN\dagger} - 8)\end{aligned}\quad (49)$$

We are interested in the last constant in order to calculate the ground state energy of the system.

4 Wave Function

We can now write down the wave function. Since θ^{MN} is the conjugate to θ^{NM} for $M < N$, we can choose θ^{MN} to be annihilation operators. Let ξ_0 represent the vacuum state vector such that the annihilation operators act as follows:

$$\theta^{MN}\xi_0 = 0 \quad (50)$$

for all indices such that $M < N$. Then the ground state vector for the fermionic variables is:

$$\xi = \left[\prod_{M < N} \prod_{\alpha}^8 (\tilde{\theta}_{-}^{MN\dagger}) \right] \xi_0 \quad (51)$$

In particular, this means that:

$$\begin{aligned} \tilde{\theta}_{+}^{MN}\xi &= 0 \\ \tilde{\theta}_{-}^{MN\dagger}\xi &= 0 \end{aligned} \quad (52)$$

With this choice, we see that:

$$H_3\xi = -8 \sum_{M < N} \omega_{MN}\xi \quad (53)$$

The total wave function can now written as:

$$\Psi = \chi(x - tV, x_a)\Phi_0(x, Y_a)\xi(x, x_a) \quad (54)$$

where t becomes large as we go along the string, and V represents the asymptotic value of the string variables, which depends on the function f .

To find total energy, we now sum the contribution of H_2 and H_3 :

$$(H_2 + H_3)\Psi = 8 \sum_{M < N} (|\lambda_{MN}| - \omega_{MN})\Psi \quad (55)$$

As before, let t represent a variable which measures how far we are along the string-like filament. We shall take $t \rightarrow \infty$. We make the assumption that x grows as t , while x_a approaches a constant. Then we see that ω_{MN}

asymptotically approaches $|\lambda_{MN}|$ in this limit, so that the ground state energy of H_2 and H_3 vanishes:

$$(H_2 + H_3) \Psi \rightarrow 0 \tag{56}$$

It is not hard to find the contribution of H_4 , which is a polynomial in Y, x, x_a . We are interested in the value of the matrix element:

$$\lim_{t \rightarrow \infty} |\Psi, P\Psi| \rightarrow t^n \tag{57}$$

for some polynomial P . Since Φ_0 is just the ground state wave function for the harmonic oscillator in terms of Y , it is easy to calculate the value of $(\Phi_0, P(Y)\Phi_0)$. We find that $n = -1/2$ for every Y contained within P . For every x contained within P , we have a contribution of $n = 1$. By simply counting the number of x and Y within H_4 , we see that it does not contribute to the ground state energy to the leading order, so it can be ignored.

In conclusion, we see that the principal contribution to the ground state energy comes from H_2 and H_3 .

Furthermore, we see that the energy eigenvalue of the operator is continuous for the ground state, which means that the system is unstable.

We caution that there may be hidden infinities with regard to our calculation. Since the continuous matrix $z(\sigma, \bar{\sigma})$ contains derivatives, it may be possible that its eigenvalues are actually divergent. Then the cancellation of the lowest eigenvalue of $H_2 + H_3$ must be carefully regularized. However, the advantage of our discussion is that it was carried out in the continuum theory rather in super Yang-Mills theory, so we have a more intuitive understanding of where the problems may arise.

5 Discussion

Although the system is unstable, we speculate how this may still be compatible with known phenomenology. For a physical system like quantum membranes to be compatible with known physics, we have to ask:

a) why don't we see them in nature?

b) do decaying fundamental particles violate unitarity or other cherished features of quantum field theory?

To answer the first objection, we note that because the decay time of such a quantum membrane is on the order of the Planck time, it is possible that unstable membranes decay too rapidly to be detected by our instruments.

One potential flaw in this argument is that there may be different values of physical parameters, such as the mass of the membrane, for which the life-time is long. Therefore, to make a rough guess of the decay life of such a quantum membrane, we recall that the decay width of the decay of a quark-anti-quark bound state is given by:

$$\Gamma = \frac{16\pi\alpha^2}{3} \frac{|\psi(0)|^2}{M^2} \quad (58)$$

where $\psi(0)$ is the wave function of the bound state at the origin, and M is the mass of the decay product. On dimensional and kinematic grounds, we expect this formula to be roughly correct for the decay of the membrane, discarding the effect of spin, quantum numbers, etc.

We expect that $|\psi(0)|$ to be on the order of a fermi⁻³, the rough size of the quark-anti-quark bound state. For our purposes, we assume that the membrane is on the order of the Planck length. On dimensional grounds, we therefore expect that the lifetime of the membrane to be on the order of:

$$T \sim M^2 L^{-3} \quad (59)$$

where L is the Planck length.

For relatively light-weight membranes, we find that the lifetime is much smaller than Planck times, so we will, as expected, never see these particles.

The other case is more interesting. For very massive membranes, we find that the lifetime becomes arbitrarily long, which seems to violate experiment. However, the coupling of very massive membranes, much heavier than the Planck mass, is very small, and hence barely couple to the particles we see in nature. Again, we see that unstable membranes cannot be measured in the laboratory.

In summary, light-weight membranes live too short to be detected, and massive (long-lived) membranes have vanishing coupling to the known particles.

Yet another objection that one may raise to our naive arguments is that membranes decay into other membranes, losing energy and mass with each decay, and hence the lifetime of the decaying membranes constantly changing. It is conceivable that, starting with a single membrane, the cascading sequence of daughter membranes may produce a collection of membranes with a lifetime long enough to be measured in the lab.

To estimate the effect of an infinite sequence of decaying membranes, let us analyze a simpler system: a chain of decaying objects, similar to the decay of a series of radio-nuclides.

Let N_i represent the amount of decaying material of type i . Let Ω_i represent the rate of decay of the i th material. Let ω_{ij} represent the rate at which substance i is decaying into substance j , which increases the amount of the j th substance. Then the coupled series of equations is given by:

$$\begin{aligned}
\dot{N}_1 &= -\Omega_1 N_1 \\
\dot{N}_2 &= -\Omega_2 N_2 + \omega_{12} N_1 \\
\dot{N}_3 &= -\Omega_3 N_3 + \omega_{13} N_1 + \omega_{23} N_2 \\
&\vdots \\
\dot{N}_j &= -\Omega_j N_j + \sum_{k=1}^{j-1} \omega_{kj} N_k
\end{aligned} \tag{60}$$

There is a simple solution to these coupled equations. If we use the ansatz:

$$\begin{aligned}
N_1 &= A_1 e^{-\Omega_1 t} \\
N_2 &= A_2 e^{-\Omega_2 t} + A_{12} e^{-\Omega_1 t} \\
N_3 &= A_3 e^{-\Omega_3 t} + A_{23} e^{-\Omega_2 t} + A_{13} e^{-\Omega_1 t} \\
&\vdots \\
N_j &= A_j e^{-\Omega_j t} + \sum_{i=1}^{j-1} A_{ij} e^{-\Omega_i t}
\end{aligned} \tag{61}$$

then the solution is given by:

$$\begin{aligned}
A_{12} &= \frac{\omega_{12}A_1}{\Omega_2 - \Omega_1} \\
A_{13} &= \frac{\omega_{13}A_1}{\Omega_3 - \Omega_1} + \frac{\omega_{23}\omega_{12}A_1}{(\Omega_2 - \Omega_1)(\Omega_3 - \Omega_1)} \\
A_{23} &= \frac{\omega_{23}\omega_{12}}{(\Omega_3 - \Omega_2)(\Omega_2 - \Omega_1)}A_2
\end{aligned} \tag{62}$$

and so on.

The lesson we learn from this is that, even with an arbitrarily large number of decaying products, each decaying into each other, the limiting factor is the longest life-time of a single decay product. The substance with the slowest decay $e^{-\Omega_i t}$ dominates the entire series.

This gives us reasonable assurance that the infinite series of decaying membranes does not behave any worse than its longest lived membrane.

And lastly, we must ask the question of whether quantum field theory can accommodate decaying fundamental particles. Previous work by McCoy and Wu^{13,14} on the field theory of decaying particles indicates that there are no fatal problems with such a theory. The pole of the two-point function corresponding to an unstable particle becomes a branch cut in this case. The principle question is whether there exists a Lehmann spectral representation of such a theory with a branch cut. In fact, two-point functions with a branch cut rather than a single pole have already been encountered in two dimensional SU(N) Yang-Mills theory in the limit $N \rightarrow \infty$, the two-dimensional SU(2) Thirring model, and two-dimensional Ising field theory.

Of course, these arguments that we have presented here are certainly not rigorous, since we do not know how membranes interact and the theory is highly non-linear. Until the interacting theory is fully calculated, we cannot know precisely whether our heuristic arguments will hold up. However, they indicate that one cannot immediately dismiss fundamental quantum membranes as a physical theory.

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